

# Endomorphisms of Exotic Models

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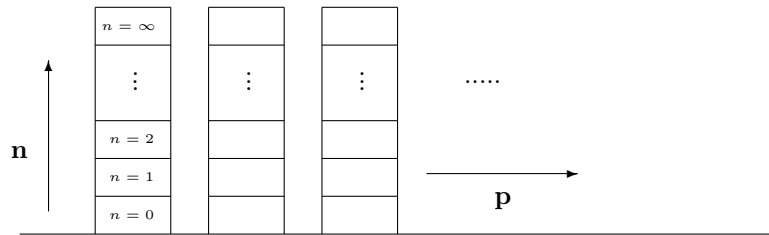
## Abstract

We calculate the endomorphism dga of Franke’s exotic algebraic model for the  $K$ -local stable homotopy category at odd primes. We unravel its original abstract structure to give explicit generators, differentials and products.

## Introduction

The stable homotopy category  $\mathrm{Ho}(\mathcal{S})$  is a large and complex category. Thus it becomes natural to break it up. First we break it into its  $p$ -local parts  $\mathrm{Ho}(\mathcal{S}_{(p)})$ , and then these are broken into smaller, atomic pieces. These pieces are given by the *chromatic localisations*  $\mathrm{Ho}(L_n\mathcal{S})$ ,  $n \in \mathbb{N}$ . (Note that the prime  $p$  is traditionally absent from notation.) We can think of the stable homotopy category as a city with a tower block with infinitely many floors for each prime, and the  $n^{\mathrm{th}}$  floor of each tower block being described by  $\mathrm{Ho}(L_n\mathcal{S})$ .

Visualising  $\mathrm{Ho}(\mathcal{S})$  in relation to  $\mathrm{Ho}(L_n\mathcal{S})$ :



The “ground floor”  $\mathrm{Ho}(L_0\mathcal{S})$  is given by rational homotopy theory; this is the same for all primes. The first floor  $\mathrm{Ho}(L_1\mathcal{S})$  is governed by  $p$ -local topological  $K$ -theory, which is related to vector bundles. The next level  $\mathrm{Ho}(L_2\mathcal{S})$  is related to elliptic curves, but is already much more complicated to describe, while the higher levels  $\mathrm{Ho}(L_n\mathcal{S})$  are valuable for their structural contribution to the bigger picture rather than any individual computational merits.

Schwede showed in [16] that the triangulated structure of  $\mathrm{Ho}(\mathcal{S})$  determines the entire higher homotopy information of spectra, that is, it determines the underlying model category up to suitable equivalence. In other words, the stable

homotopy category is *rigid*. This is particularly interesting because examples of rigidity are usually hard to find. A natural question to follow is whether the atomic building blocks  $\mathrm{Ho}(L_n\mathcal{S})$  are also rigid. Franke showed in [9] that for  $n^2 + n < 2p - 2$  (in particular for  $n = 1$  and  $p$  odd) this is false and  $\mathrm{Ho}(L_n\mathcal{S})$  are not rigid by constructing an algebraic counterexample. The second author showed in [13] that in contrast, in the case of  $n = 1$  and  $p = 2$ , the  $K$ -local stable homotopy category  $\mathrm{Ho}(L_1\mathcal{S})$  is rigid. To this day it is rather mysterious why counterexamples exist for odd primes but not for  $p = 2$ , and what the situation is like outside of the range covered by Franke and Roitzheim.

Franke's model is *algebraic*, which means that it is model enriched over the model category of chain complexes. Therefore it makes sense to direct the study of exotic models to algebraic models. For example, is Franke's model the only algebraic model for  $\mathrm{Ho}(L_1\mathcal{S})$ ? Or are all exotic models for  $\mathrm{Ho}(L_1\mathcal{S})$  algebraic?

By Morita theory, algebraic model categories are determined by an endomorphism dga with homology and Massey products. To get a grip on those uniqueness questions we have to understand the endomorphism dgas: if there was a unique endomorphism dga, then there would also be a unique algebraic model. This has partially been answered in [15] but it does not seem feasible to approach this by hand due to the rapidly increasing complexity of the computations.

Thus, in order to work towards a greater understanding of algebraic models, their uniqueness, and ultimately the stable homotopy category, we are going to look at the endomorphism dga of Franke's exotic models. This construction used many abstract ingredients such as injective resolutions of  $E(1)_*E(1)$ -comodules, Adams operations, quasi-periodicity and  $v_1$ -self maps. The goal of this paper is to carefully unravel these abstractions in order to arrive at the  $\mathbb{Z}_p$ -module structure of the dga in question. We hope that going through and turning the abstract machinery into concrete numbers will contribute to the greater picture by allowing for direct calculations in the future.

This paper is organised as follows. In Section 1 we recall some background on endomorphism dgas and the context that we are using them in. In Section 2 we give a summary of the construction and properties of Franke's exotic model for  $\mathrm{Ho}(L_1\mathcal{S})$ . In Section 3 we perform first steps to simplify the endomorphism dga of a compact generator of Franke's model. In Section 4 we show how the endomorphism dga can be expressed explicitly in terms of sequences with coefficients in  $\mathbb{Z}_p$  and  $\mathbb{Q}$ , using work of [5]. In Sections 5 and 6 we use the sequence representation to do an explicit calculation of the homology of the endomorphism dga, verifying that it gives the expected result. We conclude in Section 7 by verifying that the product and Massey products also give the expected result.

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## 1 Algebraic Models

The basic goal is to study the  $K$ -local stable homotopy category  $\mathrm{Ho}(L_1\mathcal{S})$  at an odd prime  $p$ . We assume that the reader is familiar with basic notions regarding stable model category and Bousfield localisation, see e.g. [2]. For background on  $K$ -theory and related topics, see [4]. To study  $\mathrm{Ho}(L_1\mathcal{S})$ , we will study the existence of *algebraic model categories*: a stable  $Ch(\mathbb{Z})$ -model category  $\mathcal{C}$  in the sense of [6, Appendix A], such that there is an equivalence of triangulated categories

$$\Phi : \mathrm{Ho}(L_1\mathcal{S}) \longrightarrow \mathrm{Ho}(\mathcal{C}).$$

If  $\mathcal{C}$  is an arbitrary stable model category, it can be very hard to understand it, or to compare  $L_1\mathcal{S}$  with  $\mathcal{C}$ . The following result [17, Theorem 3.1.1] gives a more concrete way to approach  $\mathcal{C}$ . Recall that an object  $X \in \mathrm{Ho}(\mathcal{C})$  is *compact* if the functor  $\mathrm{Ho}(\mathcal{C})(X, -)$  commutes with arbitrary coproducts.  $X$  is a *generator* if the full subcategory of  $\mathrm{Ho}(\mathcal{C})$  containing  $X$  which is closed under coproducts and exact triangles is again  $\mathrm{Ho}(\mathcal{C})$  itself. Then we have the following result.

**Theorem 1.** *[Schwede-Shipley] Let  $\mathcal{C}$  be a simplicial proper, stable model category with a compact generator  $X$ . Then there exists a chain of simplicial Quillen equivalences between  $\mathcal{C}$  and module spectra over the endomorphism ring spectrum of  $X$ ,*

$$\mathcal{C} \simeq \mathrm{mod}\text{-}\mathrm{End}(X).$$

Note that the assumption that  $\mathcal{C}$  is simplicial is not a significant restriction, see e.g. [7].

The category  $\mathrm{Ho}(L_1\mathcal{S})$  possesses the sphere  $L_1S^0$  as a compact generator. Thus if  $\Phi : \mathrm{Ho}(L_1\mathcal{S}) \longrightarrow \mathrm{Ho}(\mathcal{C})$  is a triangulated equivalence as above, we can use (a fibrant and cofibrant replacement of)  $X = \Phi(L_1S^0)$  as a compact generator for  $\mathrm{Ho}(\mathcal{C})$ .

From Theorem 1, we know that the endomorphism ring spectrum  $\mathrm{End}(X)$  satisfies

$$\pi_*(\mathrm{End}(X)) \cong \mathrm{Ho}(\mathcal{C})(X, X).$$

Combining this with our triangulated equivalence, we have

$$\pi_*(\mathrm{End}(X)) \cong \mathrm{Ho}(\mathcal{C})(X, X) \cong \pi_*(L_1S^0).$$

Now if we additionally assume that  $\mathcal{C}$  is an algebraic category, [8, Proposition 6.3] gives us the following about the endomorphism spectrum:

**Theorem 2.** *Let  $\mathcal{C}$  be an algebraic model category with a fibrant and cofibrant compact generator  $X$ . Then the endomorphism ring spectrum  $\mathrm{End}(X)$  is the generalised Eilenberg-Mac Lane spectrum of the endomorphism dga  $\mathcal{C}(X, X)$ .*

Moreover, for algebraic  $\mathcal{C}$  we can say that  $\mathcal{C}(X, X)$  is a differential graded algebra (dga) rather than just a graded abelian group, and the endomorphism dga  $\mathcal{C}(X, X)$  for  $X \cong \Phi(L_1 S^0)$  satisfies ([15, Lemma 2.1]):

- $H_*(\mathcal{C}(X, X)) = \text{Ho}(\mathcal{C})(X, X) = \pi_*(L_1 S^0)$ .
- Under the above, the Massey products of  $\mathcal{C}(X, X)$  coincide with the Toda brackets of  $\pi_*(L_1 S^0)$ .

Thus we see that in order to understand algebraic models  $\mathcal{C}$  for  $L_1 \mathcal{S}$  it is vital to understand the endomorphism dga of a compact generator. In the next section, we will describe a specific algebraic model  $\mathcal{C}$  that will be the focus of this paper, and also take a closer look at its compact generator.

## 2 Franke's model and its compact generator

In this section we are going to give a brief description of the particular algebraic model for  $\text{Ho}(L_1 \mathcal{S})$  that we will be looking at in detail in the subsequent sections. This was developed by Franke [9]; further details are available in [14] (and [11] for the triangulated structure). In what follows, we will use notation consistent with [14].

To begin, we consider the category  $\mathcal{B}$ , an abelian category which is equivalent to  $E(1)_* E(1)$ -comodules that are concentrated in degrees  $0 \bmod 2p - 2$ . (Note that in [4], Bousfield denotes this category by  $\mathcal{B}_*$ .) We can think of  $E(1)_* E(1)$ -comodules as modules over  $E(1)_*$  with an action of the Adams operations. Furthermore, the category  $\mathcal{B}$  is equipped with self-equivalences

$$T^{j(p-1)} : \mathcal{B} \longrightarrow \mathcal{B} \quad (j \in \mathbb{Z})$$

each of which is the identity on the underlying  $E(1)_*$ -modules but changes the Adams operation  $\Psi^k$  by a factor of  $k^{j(p-1)}$ .

Now we consider *twisted chain complexes*  $\mathcal{C}^{2p-2}(\mathcal{B})$  on  $\mathcal{B}$ . An object of  $\mathcal{C}^{2p-2}(\mathcal{B})$  is a cochain complex  $C^*$  with  $C^i \in \mathcal{B}$  together with an isomorphism

$$\alpha_C : T^{(2p-2)(p-1)}(C^*) \longrightarrow C^*[2p-2] = C^{*+2p-2}.$$

Morphisms in this category are cochain maps  $f : C^* \longrightarrow D^*$  which are compatible with those isomorphisms, i.e. for which there is a commutative diagram

$$\begin{array}{ccc} T^{(2p-2)(p-1)}(C^*) & \xrightarrow{\alpha_C} & C^*[2p-2] \\ \downarrow T^{(2p-2)(p-1)}(f) & & \downarrow f[2p-2] \\ T^{(2p-2)(p-1)}(D^*) & \xrightarrow{\alpha_D} & D^*[2p-2]. \end{array}$$

We can define a model structure on  $\mathcal{C}^{2p-2}(\mathcal{B})$  as follows.

**Proposition 3** (Franke). *There is a model structure on  $\mathcal{C}^{2p-2}(\mathcal{B})$  such that*

- *weak equivalences are the quasi-isomorphisms*
- *cofibrations are the monomorphisms*
- *fibrations are the degreewise split epimorphisms with strictly injective kernel.*

Here, an object  $C^*$  is said to be strictly injective if it is injective and for each acyclic complex  $D^*$ , the mapping chain complex  $\text{Hom}_{\mathcal{C}^{2p-2}(\mathcal{B})}(D^*, C^*)^*$  is again acyclic.

Note that the above model structure is a variant of the standard injective model structure on chain complexes. There is no projective-type model structure on  $\mathcal{C}^{2p-2}(\mathcal{B})$ , as  $\mathcal{B}$  has enough injectives but not enough projectives.

Now let  $\mathcal{D}^{2p-2}(\mathcal{B})$  be the homotopy category of a model category of  $\mathcal{C}^{2p-2}(\mathcal{B})$ . This is the exotic algebraic model we are interested in:

**Theorem 4** (Franke). *For  $p > 2$  there is an equivalence of triangulated categories*

$$\mathcal{R} : \mathcal{D}^{2p-2}(\mathcal{B}) \longrightarrow \text{Ho}(L_1\mathcal{S})$$

*which satisfies*

$$\bigoplus_{i=0}^{2p-3} H^i(C)[-i] \cong E(1)_*(\mathcal{R}(C)).$$

Now let us take a brief look at the equivalence  $\mathcal{R} : \mathcal{D}^{2p-2}(\mathcal{B}) \longrightarrow \text{Ho}(L_1\mathcal{S})$ . The notation  $\mathcal{R}$  stands for *reconstruction functor*. Usually one would expect an equivalence between two categories such as the above to have the category of topological origin as its source and the algebraic category as its target. But in this unusual case, the equivalence *reconstructs* a topological object from an algebraic one.

This reconstruction can be described as follows. To build a spectrum  $X$  from a chain complex  $C^*$ , one first considers the boundaries  $B^i$  of  $C^*$  ( $1 \leq i \leq 2p-2$ ) and the quotients  $G^i$  of  $C^*$  by its boundaries. Then, one assigns spectra  $X_{\beta_i}$  and  $X_{\gamma_i}$  to the  $B^i$  and  $G^i$  respectively, so that

$$G^i(X) = E(1)_*(X_{\gamma_i})[-i] \text{ and } B^i(X) = E(1)_*(X_{\beta_i})[-i].$$

These spectra are now arranged in a crown-shaped diagram

$$\begin{array}{ccccccc} X_{\beta_1} & & \cdots & & X_{\beta_{i-1}} & & X_{\beta_i} & & \cdots & & X_{\beta_{2p-2}} \\ \uparrow & & & & \uparrow & & \uparrow & & & & \uparrow \\ X_{\gamma_1} & & & & X_{\gamma_{i-1}} & & X_{\gamma_i} & & & & X_{\gamma_{2p-2}} \end{array}$$

(Note: In the original image, there are additional dotted arrows connecting the bottom row to the top row, forming a crown shape.)

Then the reconstruction spectrum  $X = \mathcal{R}(C^*)$  is defined to be the homotopy colimit of the above diagram. Proving that this defines an equivalence of

categories as stated in Theorem 4 is a lengthy progress involving various Adams spectral sequences and diagram chases. Once it is completed, however, it is not too hard to read off the following:

**Lemma 5.** *The cochain complex  $A^* = \mathcal{R}^{-1}(L_1 S^0)$  is  $C^i = T^{i(p-1)}(E(1)_*)$  in degrees  $i = k(2p-2), k \in \mathbb{Z}$  and 0 in all other degrees.*

□

### 3 The endomorphism dga

Recall from Section 1 that in order to understand an algebraic model, we want to study the endomorphism dga of a compact generator. We know that the cochain complex of Lemma 5

$$\begin{array}{ccccccc} A^* = & \cdots \longrightarrow & 0 \longrightarrow & T^{-(2p-2)(p-1)} E(1)_* \longrightarrow & 0 \longrightarrow & \cdots \\ & \cdots \longrightarrow & 0 \longrightarrow & E(1)_* \longrightarrow & 0 \longrightarrow & \cdots \\ & \cdots \longrightarrow & 0 \longrightarrow & T^{(2p-2)(p-1)} E(1)_* \longrightarrow & 0 \longrightarrow & \cdots \end{array}$$

is a compact generator for  $\mathcal{D}^{2p-2}(\mathcal{B})$ . Hence, to understand Franke's model we need to study the endomorphism dga of  $A^*$ .

Note that we have abstract reasons to believe that  $H^n(C^*) = \pi_n(L_1 S^0)$ : by construction,

$$H^{t-s}(C^*) = \text{Ext}_{\mathcal{B}}^{s,t-s}(E(1)_*, E(1)_*)$$

which in turn is the  $E^2$ -term of the  $E(1)_*$ -based Adams spectral sequence for  $\pi_*(L_1 S^0)$ . This spectral collapses by examining the degrees, giving the equivalence  $H^n(C^*) = \pi_n(L_1 S^0)$ . Here, we intend to unravel what  $C^*$  looks like as  $\mathbb{Z}_{(p)}$ -module and obtain a concrete description of this chain complex. We will use this to verify the homology calculation.

The first step in calculating the endomorphism dga is to find a fibrant and cofibrant replacement for  $A^*$ . The model structure of Proposition 3 implies that any object in  $\mathcal{C}^{2p-2}(\mathcal{B})$  is cofibrant, so in fact we only need a fibrant replacement.

To produce a fibrant replacement, we will use an injective resolution

$$0 \longrightarrow E(1)_* \longrightarrow I_0 \longrightarrow I_1 \longrightarrow I_2 \longrightarrow 0 \quad (1)$$

of  $E(1)_*$  as an  $E(1)_* E(1)$ -comodule. Since  $C^*$  is  $E(1)_*$  repeated periodically using the self-equivalence  $T^{(2p-2)(p-1)}$ , we will obtain an injective resolution of  $A^*$  by taking the injective resolution above and repeating it periodically, again applying the self-equivalence  $T^{(2p-2)(p-1)}$ . Since  $p$  is odd and the injective dimension of  $\mathcal{B}$  is 2 (as is the injective dimension of  $E(1)_* E(1)$ -comod) [4, Section 7], the pieces from the injective resolution do not overlap in the cochain complex.

Thus we create a fibrant replacement

$$(A^{fib})^* =$$

$$\begin{array}{ccccccccccccccc} \cdots 0 & \longrightarrow & T^{-(2p-2)(p-1)} I_0 & \longrightarrow & T^{-(2p-2)(p-1)} I_1 & \longrightarrow & T^{-(2p-2)(p-1)} I_2 & \longrightarrow & 0 & \longrightarrow & \cdots \\ \cdots 0 & \longrightarrow & & I_0 & \longrightarrow & & I_1 & \longrightarrow & & I_2 & \longrightarrow & 0 & \longrightarrow & \cdots \\ \cdots 0 & \longrightarrow & T^{(2p-2)(p-1)} I_0 & \longrightarrow & T^{(2p-2)(p-1)} I_1 & \longrightarrow & T^{(2p-2)(p-1)} I_2 & \longrightarrow & 0 & \longrightarrow & \cdots \end{array}$$

For the injective resolution in (1), we will use the standard injective resolution by Adams-Baird-Ravenel [3]

$$0 \longrightarrow E(1)_* \longrightarrow E(1)_* E(1) \xrightarrow{(\Psi^r - 1)_*} E(1)_* E(1) \xrightarrow{q} E(1)_* \otimes \mathbb{Q} \longrightarrow 0 \quad (2)$$

where  $r$  is a unit of the cyclic group  $(\mathbb{Z}/p^2)^\times$ ,  $\Psi^r$  is the  $r^{th}$  Adams operation and  $q$  is a rational isomorphism in degree 0 and trivial in other degrees.

Now the endomorphism cochain complex of two cochain complexes  $C^*$  and  $D^*$  is defined as

$$\mathrm{Hom}_{\mathcal{C}^{2p-2}(\mathcal{B})}(C^*, D^*)^n = \prod_{k \in \mathbb{Z}} \mathrm{Hom}_{\mathcal{B}}(C^n, D^{n+k})$$

[10, Chapter 4.2]. In our case, periodicity implies that the complex  $\mathrm{Hom}_{\mathcal{C}^{2p-2}(\mathcal{B})}((A^{fib})^*, (A^{fib})^*)$  is entirely determined by the terms of the form

$$\mathrm{Hom}_{\mathcal{B}}(I_j, I_k), \quad \text{where } i, j \in \{0, 1, 2\}.$$

So we have to calculate nine potential terms:

$$\begin{aligned} C^n &:= \mathrm{Hom}_{\mathcal{C}^{2p-2}(\mathcal{B})}((A^{fib})^*, (A^{fib})^*)^n = \prod_{n=t-s, i} \mathrm{Hom}_{\mathcal{B}}((I_i)_{*-t}, (I_{i+s})_*) \\ &= \begin{aligned} &\mathrm{Hom}_{\mathcal{B}}((I_0)_{*-n}, (I_0)_*) \quad \times \mathrm{Hom}_{\mathcal{B}}((I_0)_{*-(n-1)}, (I_1)_*) \quad \times \mathrm{Hom}_{\mathcal{B}}((I_0)_{*-(n-2)}, (I_2)_*) \\ &\times \mathrm{Hom}_{\mathcal{B}}((I_1)_{*-(n+1)}, (I_0)_*) \quad \times \mathrm{Hom}_{\mathcal{B}}((I_1)_{*-n}, (I_1)_*) \quad \times \mathrm{Hom}_{\mathcal{B}}((I_1)_{*-(n-1)}, (I_2)_*) \\ &\times \mathrm{Hom}_{\mathcal{B}}((I_2)_{*-(n+2)}, (I_0)_*) \quad \times \mathrm{Hom}_{\mathcal{B}}((I_2)_{*-(n+1)}, (I_1)_*) \quad \times \mathrm{Hom}_{\mathcal{B}}((I_2)_{*-n}, (I_2)_*) \end{aligned} \end{aligned}$$

and specify the differentials between those terms.

Since the terms appearing in the sequence (2) are either of the form  $E(1)_* E(1)$  or  $E(1)_* \otimes \mathbb{Q}$ , the nine terms above can be grouped into four types of the following form:

- (I)  $\mathrm{Hom}_{\mathcal{B}}(E(1)_{*-t} E(1), E(1)_* E(1))$
- (II)  $\mathrm{Hom}_{\mathcal{B}}(E(1)_{*-t} E(1), E(1)_* \otimes \mathbb{Q})$

$$(III) \quad \text{Hom}_{\mathcal{B}}(E(1)_{*-t} \otimes \mathbb{Q}, E(1)_* E(1))$$

$$(IV) \quad \text{Hom}_{\mathcal{B}}(E(1)_{*-t} \otimes \mathbb{Q}, E(1)_* \otimes \mathbb{Q})$$

All of the above are trivial unless  $t$  is a multiple of  $2p - 2$ . By [12, Appendix A1] we have the following natural isomorphism

$$\text{Hom}_{E(1)_*}(M, N) \cong \text{Hom}_{\mathcal{B}}(M, E(1)_* E(1) \otimes_{E(1)_*} N) \quad (3)$$

for  $E(1)_*$ -modules  $M$  and  $N$ . Applying this to the terms above yields the following.

**Type (I)** The isomorphism (3) gives

$$\begin{aligned} \text{Hom}_{\mathcal{B}}(E(1)_{*-t} E(1), E(1)_* E(1)) &\cong \text{Hom}_{E(1)_*}(E(1)_{*-t} E(1), E(1)_*) \\ &\cong \text{Hom}_{\mathbb{Z}_{(p)}}(E(1)_0 E(1), \mathbb{Z}_{(p)}(v_1^k)) \quad \text{for } t = (2p - 2)k \end{aligned}$$

**Type (II)** Similarly, since  $E(1)_0 E(1)$  is free over  $E(1)_0$ , here the isomorphism (3) gives

$$\begin{aligned} \text{Hom}_{\mathcal{B}}(E(1)_{*-t} E(1), E(1)_* \otimes \mathbb{Q}) &\cong \text{Hom}_{E(1)_*}(E(1)_{*-t} E(1), \mathbb{Q}) \\ &\cong \text{Hom}_{\mathbb{Z}_{(p)}}(E(1)_0 E(1), \mathbb{Q}\{v_1^k\}) \quad \text{for } t = (2p - 2)k \end{aligned}$$

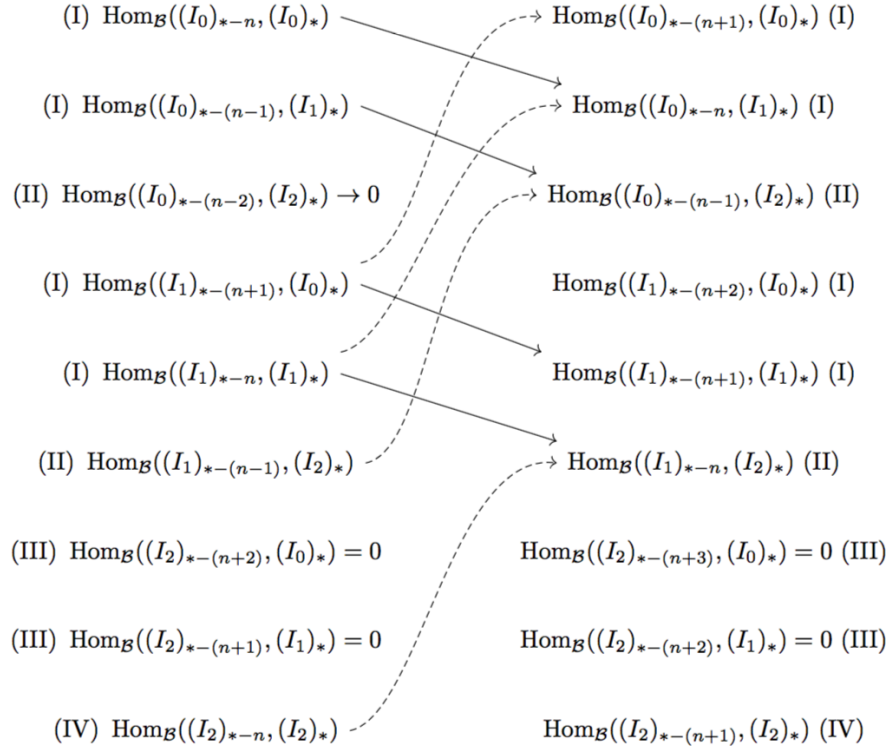
**Type (III)** Every  $E(1)_*$ -module is in particular a  $\mathbb{Z}_{(p)}$ -module, and so every element of (III) is in particular a  $\mathbb{Z}_{(p)}$ -module homomorphism from  $\mathbb{Q}$  to  $\mathbb{Z}_{(p)}$ . Thus, all terms of the form (III) are zero.

**Type (IV)** Here we have

$$\begin{aligned} \text{Hom}_{\mathcal{B}}(E(1)_{*-t} \otimes \mathbb{Q}, E(1)_* \otimes \mathbb{Q}) &\cong \text{Hom}_{E(1)_*}(E(1)_{*-t} \otimes \mathbb{Q}, \mathbb{Q}) \\ &= \text{Hom}_{E(1)_*}(\mathbb{Q}\{v_1^{-k}\}, \mathbb{Q}[0]) \\ &= \begin{cases} \mathbb{Q} & \text{for } t = (2p - 2)k = 0 \\ 0 & \text{else} \end{cases} \end{aligned}$$

Next, we want to consider the differentials in the endomorphism complex. A differential from  $C^n$  to  $C^{n+1}$  is of the form  $d \circ f + (-1)^{n+1} f \circ d$ . We illustrate its individual parts in the diagram below, where a solid arrow represents a possible nontrivial  $d \circ f$  and a dashed arrow represents a possible nontrivial  $f \circ d$ . In addition, each term has been labeled with its type (I-IV).





Combining this information with the interpretations of Terms (I)-(IV), we see that when  $n = (2p - 2)k$ , the nontrivial parts of the dga look like:

$$C^{(2p-2)k-1} \longrightarrow C^{(2p-2)k} \longrightarrow C^{(2p-2)k+1} \longrightarrow C^{(2p-2)k+2}$$

$$\begin{array}{ccccc}
& & \mathrm{Hom}_{\mathcal{B}}(E(1)_{*-n}E(1), E(1)_*E(1)) & & \\
& \nearrow \Psi^* & & \searrow \Psi_* & \\
\mathrm{Hom}_{\mathcal{B}}(E(1)_{*-n}E(1), E(1)_*E(1)) & & & & \mathrm{Hom}_{\mathcal{B}}(E(1)_{*-n}E(1), E(1)_*E(1)) \\
& \searrow \Psi_* & \nearrow \Psi^* & & \searrow q_* \\
& & \mathrm{Hom}_{\mathcal{B}}(E(1)_{*-n}E(1), E(1)_*E(1)) & & \mathrm{Hom}_{\mathcal{B}}(E(1)_{*-n}E(1), E(1)_* \otimes \mathbb{Q}) \\
& & \searrow q_* & \nearrow \Psi^* & \\
& & & & \mathrm{Hom}_{\mathcal{B}}(E(1)_{*-n}E(1), E(1)_* \otimes \mathbb{Q}) \\
& \nearrow q^* & & & \\
& & \mathrm{Hom}_{\mathcal{B}}(E(1)_* \otimes \mathbb{Q}, E(1)_* \otimes \mathbb{Q}) & & 
\end{array}$$

Here,  $\Psi^*$ ,  $\Psi_*$ ,  $q^*$  and  $q_*$  refer to (pre)composing with  $\Psi = (\Psi^r - 1)$  and  $q$  from the Adams-Ravenel-Baird resolution (2).

Note that when  $k \neq 0$ , the bottom Type (IV) term  $\mathrm{Hom}_{\mathcal{B}}(E(1)_* \otimes \mathbb{Q}, E(1)_* \otimes \mathbb{Q})$  is 0 and can be ignored.

## 4 Reinterpretation as Sequences

We now turn to creating an explicit description of the sequence described in the previous section. As noted above, terms of Type (III) are trivial and terms of Type (IV) contribute only a single  $\mathbb{Q}$  in degree 0. We here consider the other, not so simple terms.

We start by considering terms of type (I). As mentioned above, by [12, Appendix A1],

$$\mathrm{Hom}_{\mathcal{B}}(E(1)_{*-n}E(1), E(1)_*E(1)) \cong \mathrm{Hom}_{E(1)_*}(E(1)_*E(1), E(1)_{*+n}). \quad (4)$$

Since  $E(1)_*E(1)$  is free as an  $E(1)_*$ -module [1, Theorem 2.1],

$$\mathrm{Hom}_{E(1)_*}(E(1)_*E(1), E(1)_{*+n}) \simeq \mathrm{Hom}_{\mathbb{Z}_{(p)}}(E(1)_0E(1), \mathbb{Z}_{(p)}[v_1^k]), \quad n = (2p-2)k.$$

This dual has been considered in [5], where it is shown that

$$\mathrm{Hom}_{\mathbb{Z}_{(p)}}(E(1)_0E(1), \mathbb{Z}_{(p)}[v_1^k]) \cong E(1)^0E(1).$$

Furthermore, by [5, Theorem 6.2] this can be uniquely expressed as a formal series

$$E(1)^0E(1) \cong \left\{ \sum_{n \geq 0} a_n \Theta_n(\Psi^r) \mid a_n \in \mathbb{Z}_{(p)} \right\}$$

see also [18, Proposition 18]. Here,  $\Theta_m$  is an explicit polynomial in the Adams operation  $\Psi^r$  (where  $r$  a generator of  $(\mathbb{Z}/p^2)^\times$ ) defined as follows: [5, Definition 6.1]:

$$\begin{aligned} \Theta_0(\Psi^r) &= 1, \\ \Theta_1(\Psi^r) &= (\Psi^r - 1), \\ \Theta_2(\Psi^r) &= (\Psi^r - 1)(\Psi^r - r), \\ \Theta_3(\Psi^r) &= (\Psi^r - 1)(\Psi^r - r)(\Psi^r - r^{-1}), \\ \Theta_4(\Psi^r) &= (\Psi^r - 1)(\Psi^r - r)(\Psi^r - r^{-1})(\Psi^r - r^2), \\ \Theta_5(\Psi^r) &= (\Psi^r - 1)(\Psi^r - r)(\Psi^r - r^{-1})(\Psi^r - r^2)(\Psi^r - r^{-2}) \\ &\text{etc.} \end{aligned}$$

This means that we can view the elements of  $\mathrm{Hom}_{\mathcal{B}}(E(1)_{*-n}E(1), E(1)_*E(1))$  as sequences of coefficients in  $p$ -local integers,

$$\mathrm{Hom}_{\mathcal{B}}(E(1)_{*-n}E(1), E(1)_*E(1)) \cong \{(a_m)_{m \in \mathbb{N}} \mid a_m \in \mathbb{Z}_{(p)}\} = \mathbb{Z}_{(p)}^{\mathbb{N}}.$$

By the same process, we can consider terms of Type (II) as sequences of coefficients in  $\mathbb{Q}$ :

$$\mathrm{Hom}_{\mathcal{B}}(E(1)_{*-n}E(1), E(1)_* \otimes \mathbb{Q}) \cong \{(a_m)_{m \in \mathbb{N}} \mid a_m \in \mathbb{Q}\} = \mathbb{Q}^{\mathbb{N}}.$$

For simplicity of notation, will denote a sequence  $(a_m)_{m \in \mathbb{N}}$  by  $\langle a_m \rangle$  (where  $a_m$  is either in  $\mathbb{Z}_{(p)}$  or  $\mathbb{Q}$ ).

#### 4.1 The formulas on sequences

To get the differential, we need to translate the following maps over to the sequence representation:

$$\begin{aligned} \Psi^* : \mathbb{Z}_{(p)}^{\mathbb{N}} &\longrightarrow \mathbb{Z}_{(p)}^{\mathbb{N}} \\ \Psi_* : \mathbb{Z}_{(p)}^{\mathbb{N}} &\longrightarrow \mathbb{Z}_{(p)}^{\mathbb{N}} \\ \Psi^* : \mathbb{Q}^{\mathbb{N}} &\longrightarrow \mathbb{Q}^{\mathbb{N}} \\ q_* : \mathbb{Z}_{(p)}^{\mathbb{N}} &\longrightarrow \mathbb{Q}^{\mathbb{N}} \end{aligned}$$

**The map  $\Psi_*$ :** We start by considering the map  $\Psi_* = (\Psi^r - 1)_*$  given by composition with the map  $\Psi^r - 1$ . Chasing through our equivalences, we have

$$\begin{array}{ccc}
\mathrm{Hom}_{E(1)_*E(1)}(E(1)_*E(1), E(1)_*E(1))_t & \xrightarrow{(\Psi^r - 1)_*} & \mathrm{Hom}_{E(1)_*E(1)}(E(1)_*E(1), E(1)_*E(1))_t \\
\downarrow \simeq & & \downarrow \simeq \\
\mathrm{Hom}_{E(1)_*}(E(1)_*E(1), E(1)_*)_t & \xrightarrow{(\Psi^r - 1)_*} & \mathrm{Hom}_{E(1)_*}(E(1)_*E(1), E(1)_*)_t \\
\downarrow \simeq & & \downarrow \simeq \\
\mathrm{Hom}_{\mathbb{Z}_{(p)}}(E(1)_0E(1), \mathbb{Z}_{(p)}[v_1^k]) & \xrightarrow{\quad} & \mathrm{Hom}_{\mathbb{Z}_{(p)}}(E(1)_0E(1), \mathbb{Z}_{(p)}[v_1^k]) \\
\downarrow \simeq & & \downarrow \simeq \\
E(1)^0E(1) & \xrightarrow{\quad} & E(1)^0E(1) \\
\downarrow \simeq & & \downarrow \simeq \\
\{ \sum_{m \geq 0} a_m \Theta_m(\Psi^r) \mid a_m \in \mathbb{Z}_{(p)} \} & \xrightarrow{\quad} & \{ \sum_{m \geq 0} a_m \Theta_m(\Psi^r) \mid a_m \in \mathbb{Z}_{(p)} \}
\end{array}$$

To calculate  $(\Psi^r - 1)_*$  we can work on the  $v_1^k$  level. Note that when  $k = 0$ ,  $\Psi^r$  acts as the identity, and so  $(\Psi^r - 1)_* = 0$ . For  $k \neq 0$ , we know that up to a  $p$ -local unit,

$$\begin{aligned}
(\Psi^r - 1)_*(v_1^k) &= (r^{k(p-1)} - 1)v_1^k \\
&= p^{\nu(k)+1}v_1^k
\end{aligned}$$

Therefore we can see that  $(\Psi^r - 1)_*$  is given by multiplication by  $p^{\nu(k)+1}$ .

**The map  $\Psi^*$ :** Chasing through the effect of  $(\Psi^r - 1)^*$  is slightly more involved. Starting with the  $k = 0$  case, we see that since the vertical isomorphisms in the last step are ring isomorphisms, the overall effect on the sequences is multiplication by  $\Theta_1(\Psi^r)$ . In all that follows, we will write  $\Theta_i$  in place of  $\Theta_i(\Psi^r)$ . Then we can calculate:

$$\begin{aligned}
\Theta_0\Theta_1 &= \Theta_1 \\
\Theta_m\Theta_1 &= (\Psi^r - 1)(\Psi^r - r)(\Psi^r - r^{-1}) \cdots (\Psi^r - r^{\tilde{s}(m)})(\Psi^r - 1) \\
\Theta_{m+1} &= (\Psi^r - 1)(\Psi^r - r)(\Psi^r - r^{-1}) \cdots (\Psi^r - r^{\tilde{s}(m)})(\Psi^r - r^{\tilde{s}(m+1)})
\end{aligned}$$

where

$$\tilde{s}(m) = \begin{cases} \frac{m}{2} & m \text{ even} \\ \frac{1-m}{2} & m \text{ odd} \end{cases}$$

So then

$$\begin{aligned}\Theta_m \Theta_1 - \Theta_{m+1} &= (\Psi^r - 1)(\Psi^r - r)(\Psi^r - r^{-1}) \cdots (\Psi^r - r^{\tilde{s}(m)})(r^{\tilde{s}(m+1)} - 1) \\ \Theta_m \Theta_1 &= [r^{\tilde{s}(m+1)} - 1] \Theta_m + \Theta_{m+1}\end{aligned}$$

Therefore

$$\sum_{m \geq 0} a_m \Theta_m \Theta_1 = a_0 \Theta_1 + \sum_{m \geq 1} a_m [r^{s(m)} - 1] \Theta_m + \Theta_{m+1} = \sum_{m \geq 1} (a_m (r^{s(m)} - 1) + a_{m-1}) \Theta_m$$

where  $s(m) = \tilde{s}(m+1)$ . Thus when  $k = 0$ , our formula becomes

$$\Psi^* \langle a_m \rangle = \begin{pmatrix} 0 \\ a_1(r^{s(1)} - 1) + a_0 \\ a_2(r^{s(2)} - 1) + a_1 \\ \vdots \\ a_m(r^{s(m)} - 1) + a_{m-1} \\ \vdots \end{pmatrix}$$

When  $k \neq 0$ , then we have  $n \neq 0$  and thus, must determine how the map  $\Psi^r - 1$  behaves on  $E(1)_{*-n}E(1)$  instead of just  $E(1)_*E(1)$ . Do to this, we observe what happens on the level of the generators  $v_1^i$ . We first note that

$$(\Psi^r - 1)v_1^i = (r^{i(p-1)} - 1)v_1^i.$$

As mentioned above, precomposing with such a map corresponds to multiplication by  $\Theta_1$ . Thus, upon shifting to  $E(1)_{*-n}E(1)$  via multiplication by  $v_1^k$ , we see that multiplication by  $\Theta_1$  would correspond to  $\Psi^r - 1$  producing  $(r^{i(p-1)} - 1)v_1^{i+k}$  in  $E(1)_{*-n}E(1)$ . However, to truly shift to working in  $E(1)_{*-n}E(1)$  we observe that

$$(\Psi^r - 1)v_1^{i+k} = (r^{(i+k)(p-1)} - 1)v_1^{i+k}.$$

Due to this difference, precomposition with  $\Psi^r - 1$  on  $E(1)_{*-n}E(1)$  when translated to sums of  $\Theta_m$ 's must include an additional  $\Theta_0$  term. Up to a  $p$ -local unit, for any  $i$ ,  $r^{(i+k)(p-1)} - r^{i(p-1)} = p^{\nu(k)+1}$ . Hence, when  $k \neq 0$ ,  $\Psi^*$  acts by multiplication by  $\Theta_1 + p^{\nu(k)+1}\Theta_0$ . By performing a similar computation to the one above for  $\sum_{m \geq 0} a_m \Theta_m \Theta_1$ , we obtain

$$\sum_{m \geq 0} a_m \Theta_m (\Theta_1 + p^{\nu(k)+1}\Theta_0) = p^{\nu(k)+1}a_0 + \sum_{m \geq 1} a_m [r^{s(m)} - 1 + p^{\nu(k)+1}] \Theta_m$$

and thus, when  $k \neq 0$  our formula becomes

$$\Psi^* \langle a_m \rangle = \left\langle \begin{array}{c} p^{\nu(k)+1} a_0 \\ a_1(r^{s(1)} - 1 + p^{\nu(k)+1}) + a_0 \\ \vdots \\ a_m(r^{s(m)} - 1 + p^{\nu(k)+1}) + a_{m-1} \\ \vdots \end{array} \right\rangle$$

**The map  $q_*$ :** The map

$$q_* : \text{Hom}_{\mathcal{B}}(E(1)_{*-n}E(1), E(1)_*E(1)) \longrightarrow \text{Hom}_{\mathcal{B}}(E(1)_{*-n}E(1), E(1)_* \otimes \mathbb{Q})$$

is the map obtained by composing with the map

$$q : E(1)_*E(1) \longrightarrow E(1)_* \otimes \mathbb{Q}$$

from the Adams-Baird-Ravenel resolution. The map  $q$  is induced by the map  $E(1) \rightarrow H\mathbb{Q}$  which, on homotopy, is a rational isomorphism in degree 0 and trivial in all other degrees. So when  $n = 0$ , this map comes from the inclusion  $\mathbb{Z}_{(p)} \hookrightarrow \mathbb{Q}$  and becomes  $q_*(\langle a_m \rangle) = \langle a_m \rangle$ , and when  $n \neq 0$  this map is identically 0.

**The map  $q^*$ :** Lastly we consider the map  $q^*$  from  $\text{Hom}_{\mathcal{B}}(E(1)_* \otimes \mathbb{Q}, E(1)_* \otimes \mathbb{Q})$  to  $\text{Hom}_{\mathcal{B}}(E(1)_{*-n}E(1), E(1)_* \otimes \mathbb{Q})$ . When  $n \neq 0$ ,  $\text{Hom}_{\mathcal{B}}(E(1)_* \otimes \mathbb{Q}, E(1)_* \otimes \mathbb{Q}) = 0$ . When  $n = 0$ , we have the map  $\mathbb{Q} \rightarrow \mathbb{Q}^{\mathbb{N}}$  given by

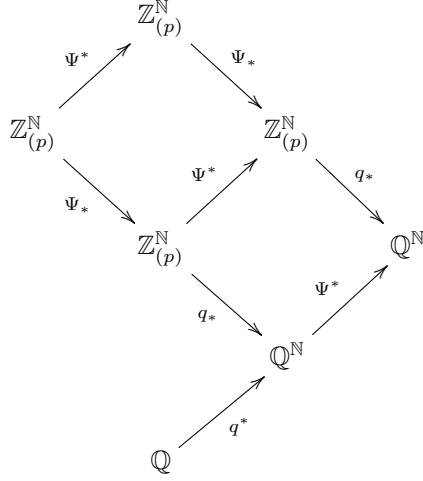
$$q^*(x) = \left\langle \begin{array}{c} x \\ 0 \\ 0 \\ \vdots \end{array} \right\rangle$$

## 5 The calculation for $n = 0$

In this section and the next, we are going to use our explicit representations to calculate the homology of the endomorphism dga  $C$ . Note that by our earlier remarks, we know that this should come out to  $H^*(C) = \pi_{-*}(L_1 S^0)$  (the change in sign arises as the dga is cohomologically graded).

As explained at the end of Section 3, in degrees around 0 our dga looks like

$$C^{-1} \xrightarrow{d^{-1}} C^0 \xrightarrow{d^0} C^1 \xrightarrow{d^1} C^2$$



Condensing it down we have

$$0 \longrightarrow \mathbb{Z}_{(p)}^{\mathbb{N}} \xrightarrow{d^{-1}} \mathbb{Z}_{(p)}^{\mathbb{N}} \oplus \mathbb{Z}_{(p)}^{\mathbb{N}} \oplus \mathbb{Q} \xrightarrow{d^0} \mathbb{Z}_{(p)}^{\mathbb{N}} \oplus \mathbb{Q}^{\mathbb{N}} \xrightarrow{d^1} \mathbb{Q}^{\mathbb{N}} \longrightarrow 0$$

-1

0

1

2

Now we have the image of  $\mathbb{Q}$  appearing in  $\mathbb{Q}^{\mathbb{N}}$ . It will be convenient for us to consider these sequences to have been ‘twisted’ by a rational number  $x \in \mathbb{Q}$ , where  $x$  is the image of  $q^*$ . In order to use this language, we introduce the following: define

$$\epsilon : \mathbb{Q}^{\mathbb{N}} \oplus \mathbb{Q} \rightarrow \mathbb{Q}^{\mathbb{N}} \oplus \mathbb{Q} \quad \epsilon(\langle a_m \rangle, x) = \left( \left\langle \begin{array}{c} a_0 + x \\ a_1 \\ a_2 \\ \vdots \end{array} \right\rangle, -x \right)$$

and observe  $\epsilon^2 = \text{id}$ . We write  $\langle a_m \rangle_x = \epsilon(\langle a_m \rangle, x)$  and we say that  $\langle a_m \rangle_x$  is the sequence  $\langle a_m \rangle$  twisted by  $x$ . We will need to track the twisting, and so we include an extra  $\mathbb{Q}$  in degree 1 to remember this twisting element. Thus we are actually looking at

$$0 \longrightarrow \mathbb{Z}_{(p)}^{\mathbb{N}} \xrightarrow{d^{-1}} \mathbb{Z}_{(p)}^{\mathbb{N}} \oplus \mathbb{Z}_{(p)}^{\mathbb{N}} \oplus \mathbb{Q} \xrightarrow{d^0} \mathbb{Z}_{(p)}^{\mathbb{N}} \oplus \mathbb{Q}^{\mathbb{N}} \oplus \mathbb{Q} \xrightarrow{d^1} \mathbb{Q}^{\mathbb{N}} \longrightarrow 0$$

-1

0

1

2

With this notation, we have that the map  $q^*(x)$  is just the zero sequence twisted by the rational number  $x$ ; once we track the twisting, we have the formula

$$q^*(x) = \langle 0 \rangle_x = \epsilon(0, x) = \left( \left\langle \begin{array}{c} x \\ 0 \\ 0 \\ \vdots \end{array} \right\rangle, -x \right)$$

Now we need to see what each of these maps is on sequences. Using our formulas from Section 4.1 we get

$$d^{-1}(\langle a_m \rangle) = (\Psi^* \langle a_m \rangle, \Psi_* \langle a_m \rangle, 0) = \left( \left\langle \begin{array}{c} 0 \\ a_1(r^{s(1)} - 1) + a_0 \\ \vdots \\ a_m(r^{s(m)} - 1) + a_{m-1} \\ \vdots \end{array} \right\rangle, \left\langle \begin{array}{c} 0 \\ 0 \\ \vdots \\ 0 \\ \vdots \end{array} \right\rangle, 0 \right)$$

The map  $d^0$  is defined by

$$\begin{aligned} d^0(\langle a_m \rangle, \langle b_m \rangle, x) &= (\Psi_* \langle a_m \rangle - \Psi^* \langle b_m \rangle, q_* \langle b_m \rangle - q^*(x)) \\ &= \left( \left\langle \begin{array}{c} 0 \\ -b_1(r^{s(1)} - 1) - b_0 \\ \vdots \\ -b_m(r^{s(m)} - 1) - b_{m-1} \\ \vdots \end{array} \right\rangle, \left\langle \begin{array}{c} b_0 - x \\ b_1 \\ \vdots \\ b_m \\ \vdots \end{array} \right\rangle, x \right) \end{aligned}$$

where the last integer is tracking our twisting as discussed above.

Now we consider the formula for  $d^1(\langle a_m \rangle, \langle b_m \rangle, y)$ . Note that the  $y$  denotes our tracking of the twisting element, and thus we are actually calculating

$$d^1(\langle a_m \rangle, \langle b_m \rangle_y). \text{ Since } \langle b_m \rangle_y = \epsilon(\langle b_m \rangle, y) = \left( \left\langle \begin{array}{c} b_0 + y \\ b_1 \\ b_2 \\ \vdots \end{array} \right\rangle, -y \right) \text{ and } \Psi^* \text{ acts}$$

only on the actual sequence, we obtain the formula

$$d^1(\langle a_m \rangle, \langle b_m \rangle, y) = q_* \langle a_m \rangle + \Psi^*(\langle b_m \rangle_y) = \left\langle \begin{array}{c} a_0 \\ a_1 + b_1(r^{s(1)} - 1) + b_0 + y \\ \vdots \\ a_m + b_m(r^{s(m)} - 1) + b_{m-1} \\ \vdots \end{array} \right\rangle$$

**Lemma 6.** *The sequences of maps  $d^{-1}$ ,  $d^0$  and  $d^{-1}$  give a cochain complex.*



*Proof.* It is easy to see that  $(d^0 \circ d^{-1})(a_m) = d^0(\Psi^*(a_m), 0, 0) = 0$  and

$$\begin{aligned}
(d^1 \circ d^0)(\langle a_m \rangle, \langle b_m \rangle, x) &= d^1 \left( \left\langle \begin{array}{c} 0 \\ -b_1(r^{s(1)} - 1) - b_0 \\ -b_2(r^{s(2)} - 1) - b_1 \\ \vdots \\ -b_m(r^{s(m)} - 1) - b_{m-1} \\ \vdots \end{array} \right\rangle, \left\langle \begin{array}{c} 0 \\ b_0 - x \\ b_1 \\ \vdots \\ b_m \\ \vdots \end{array} \right\rangle, x \right) \\
&= \left\langle \begin{array}{c} 0 \\ (-b_1(r^{s(1)} - 1) - b_0) + (b_1(r^{s(1)} - 1) + (b_0 - x) + x) \\ (-b_2(r^{s(2)} - 1) - b_1) + (b_2(r^{s(2)} - 1) + b_1) \\ \vdots \\ (-b_m(r^{s(m)} - 1) - b_{m-1}) + (b_m(r^{s(m)} - 1) + b_{m-1}) \\ \vdots \end{array} \right\rangle \\
&= 0
\end{aligned}$$

□

**Theorem 7.** Near  $n = 0$ :

$$H^n(C) = \begin{cases} 0 & \text{if } n = -1, \\ \mathbb{Z}_{(p)} & \text{if } n = 0 \\ 0 & \text{if } n = 1, \\ \mathbb{Q}/\mathbb{Z}_{(p)} & \text{if } n = 2 \end{cases}$$

*Proof.*  **$n = -1$ :** Suppose that  $\langle a_m \rangle \in \ker(d^{-1})$ . Then we know that  $a_m(r^{s(m)} - 1) + a_{m-1} = 0$  for all  $m \in \mathbb{N}$ . We will show that  $a_m = 0$  for all  $m \in \mathbb{N}$ . For any given  $m$ , choose  $\ell > m$  such that  $p(p-1)|s(\ell)$ . Then  $r^{s(\ell)} = 1$ , and so we know that  $a_{\ell-1} = 0$ . Then since  $a_m(r^{s(m)} - 1) + a_{m-1} = a_{m-1} = 0$ , we see that if  $a_j = 0$  then  $a_{j-1} = 0$  also. So by induction,  $a_m = 0$  also. So  $d^{-1}$  is injective and  $H^{-1}(C) = 0$ .

**$n = 0$ :** Suppose that  $(\langle a_m \rangle, \langle b_m \rangle, x) \in \ker d^0$ . Then we know that

$$\left( \left\langle \begin{array}{c} 0 \\ -b_1(r^{s(1)} - 1) - b_0 \\ \vdots \\ -b_m(r^{s(m)} - 1) - b_{m-1} \\ \vdots \end{array} \right\rangle, \left\langle \begin{array}{c} b_0 - x \\ b_1 \\ \vdots \\ b_m \\ \vdots \end{array} \right\rangle, x \right) = 0$$

Therefore  $x = 0$  and  $b_m = 0$ , and  $(\langle a_m \rangle, \langle b_m \rangle, x) = (\langle a_m \rangle, \langle 0 \rangle, 0)$ .

We claim that  $\text{Im } d^{-1} = (\langle a_m \rangle, \langle 0 \rangle, 0)$  where  $a_0 = 0$ . It is clear that any element in the image must have  $a_0 = 0$ . Conversely, given  $\langle a_m \rangle$  with  $a_0 = 0$ ,

we can produce  $\langle c_m \rangle$  such that  $d^{-1}(\langle c_m \rangle) = (\langle a_m \rangle, \langle 0 \rangle, 0)$ . We produce  $c_m$  as follows: again, for any  $m$ , we choose the smallest  $\ell > m$  such that  $(p-1)p|\ell$ . Then we need to choose  $c_{\ell-1}$  satisfying  $c_{\ell-1} = a_\ell$ . Then we work our way down, observing that if we have chosen  $c_j$ , we can then find  $c_{j-1}$  to satisfy  $c_j(r^{s(j)} - 1) + c_{j-1} = a_{j-1}$ . Inductively we can get a value for  $c_m$ .

So we can find  $\langle c_m \rangle$  such that  $c_m = a_m(r^{s(m)} - 1) + a_{m-1}$  and hence  $d^{-1}(\langle c_m \rangle) = (\langle a_m \rangle, \langle 0 \rangle, 0)$ . Thus we see that  $\ker d^0 / \text{Im } d^{-1} = \mathbb{Z}_{(p)}$  as represented by the value of  $a_0$  in  $(\langle a_m \rangle, \langle 0 \rangle, 0)$ .

**$n = 1$ :**  $\ker d^1 = (\langle a_m \rangle, \langle b_m \rangle, y)$  such that

$$\begin{aligned} a_0 &= 0 \\ a_1 + b_1(r^{s(1)} - 1) + b_0 + y &= 0 \\ a_2 + b_2(r^{s(2)} - 1) + b_1 &= 0 \\ a_m + b_m(r^{s(m)} - 1) + b_{m-1} &= 0 \end{aligned}$$

Now a priori we have  $a_m \in \mathbb{Z}_{(p)}$  and  $b_m, y \in \mathbb{Q}$ . However, we note that in fact we must have  $b_m \in \mathbb{Z}_{(p)}$  for  $m \geq 1$ , and  $b_0 + y_0 \in \mathbb{Z}_{(p)}$ : again if we fix  $m$  and choose  $\ell > m$  such that  $(p-1)p|\ell$ , then we have  $r^{s(\ell)} - 1 = 0$  and so  $a_\ell + b_{\ell-1} = 0$ . Hence  $b_{\ell-1} \in \mathbb{Z}_{(p)}$ ; inducting downwards shows that  $b_m \in \mathbb{Z}_{(p)}$  and at the bottom  $b_0 + y \in \mathbb{Z}_{(p)}$ .

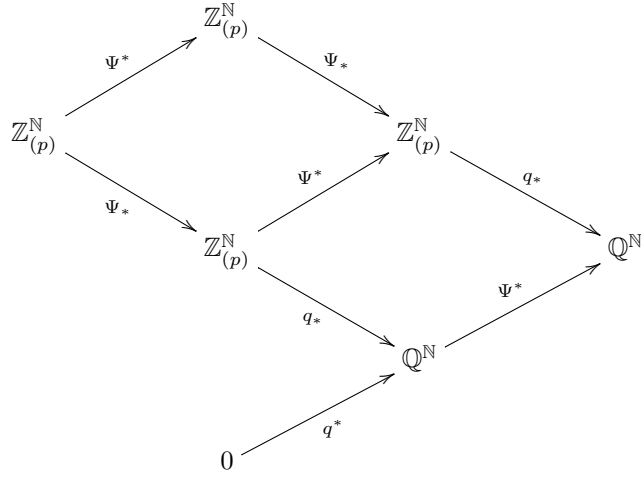
This allows us to show that there exists  $(\langle c_m \rangle, \langle d_m \rangle, x) \in \mathbb{Z}_{(p)}^{\mathbb{N}} \oplus \mathbb{Z}_{(p)}^{\mathbb{N}} \oplus \mathbb{Q}$  such that  $d^0(\langle c_m \rangle, \langle d_m \rangle, x) = (\langle a_m \rangle, \langle b_m \rangle, y)$ : we can choose  $y = 0$ , any values in  $\mathbb{Z}_{(p)}$  for  $b_m$ , and set  $a_1 = -(b_1(r^{s(1)} - 1) + b_0 + y)$ , and  $a_m = -(b_m(r^{s(m)} - 1) + b_{m-1})$ .

**$n = 2$ :** Finally, we know that  $\ker d^2 = \mathbb{Q}^{\mathbb{N}}$ . Clearly any  $\langle c_m \rangle \in \text{Im } d^1$  must have  $c_0 \in \mathbb{Z}_{(p)}$ ; we claim that the converse is true: if  $c_0 \in \mathbb{Z}_{(p)}$  then we can find  $(\langle a_m \rangle, \langle b_m \rangle, y)$  such that  $d^1(\langle a_m \rangle, \langle b_m \rangle, y) = \langle c_m \rangle$ . To do this, we may choose  $a_0 = c_0, b_0 = 0, y = 0$ , and  $a_m = 0$  for  $m \geq 1$ ; then we can set  $b_m = \frac{c_m}{r^{s(m)} - 1} \in \mathbb{Q}$  to get the desired sequences. So  $H^2(C) = \mathbb{Q}/\mathbb{Z}_{(p)}$  represented by the value of  $c_0$ .  $\square$

## 6 The Homology Calculation for $n \neq 0$

Looking back on our description of the endomorphism dga at the end of Section 3, we see that in terms of our sequence representations, we have

$$C^{(2p-2)k-1} \longrightarrow C^{(2p-2)k} \longrightarrow C^{(2p-2)k+1} \longrightarrow C^{(2p-2)k+2}$$



Condensing down our earlier diagram, we are looking at

$$0 \longrightarrow \mathbb{Z}_{(p)}^N \xrightarrow{d^{(2p-2)k-1}} \mathbb{Z}_{(p)}^N \oplus \mathbb{Z}_{(p)}^N \xrightarrow{d^{(2p-2)k}} \mathbb{Z}_{(p)}^N \oplus \mathbb{Q}^N \xrightarrow{d^{(2p-2)k+1}} \mathbb{Q}^N \longrightarrow 0$$

(2p-2)k-1

(2p-2)k

(2p-2)k+1

(2p-2)k+2

(5)

where

$$d^{(2p-2)k-1}(\langle a_m \rangle) = (\Psi^* \langle a_m \rangle, \Psi_* \langle a_m \rangle)$$

$$= \left( \left\langle \begin{array}{c} p^{\nu(k)+1} a_0 \\ a_1(r^{s(1)} - 1 + p^{\nu(k)+1}) + a_0 \\ \vdots \\ a_m(r^{s(m)} - 1 + p^{\nu(k)+1}) + a_{m-1} \\ \vdots \end{array} \right\rangle, \left\langle \begin{array}{c} p^{\nu(k)+1} a_0 \\ p^{\nu(k)+1} a_1 \\ \vdots \\ p^{\nu(k)+1} a_m \\ \vdots \end{array} \right\rangle \right)$$

$$\begin{aligned}
d^{(2p-2)k}(\langle a_m \rangle, \langle b_m \rangle) &= (\Psi_* \langle a_m \rangle - \Psi^*(\langle b_m \rangle), 0) \\
&= \left\langle \begin{pmatrix} p^{\nu(k)+1}a_0 - p^{\nu(k)+1}b_0 \\ p^{\nu(k)+1}a_1 - b_1(r^{s(1)} - 1 + p^{\nu(k)+1}) - b_0 \\ \vdots \\ p^{\nu(k)+1}a_m - b_m(r^{s(m)} - 1 + p^{\nu(k)+1}) - b_{m-1} \\ \vdots \end{pmatrix} \right\rangle, \left\langle \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \vdots \end{pmatrix} \right\rangle \\
d^{(2p-2)k+1}(\langle a_m \rangle, \langle b_m \rangle) &= \Psi^*(\langle b_m \rangle) = \left\langle \begin{pmatrix} p^{\nu(k)+1}b_0 \\ b_1(r^{s(1)} - 1 + p^{\nu(k)+1}) + b_0 \\ \vdots \\ b_m(r^{s(m)} - 1 + p^{\nu(k)+1}) + b_{m-1} \\ \vdots \end{pmatrix} \right\rangle
\end{aligned}$$

We start by verifying the following:

**Lemma 8.** *The sequence of modules and maps described in (5) is a cochain complex.*

*Proof.* **(2p - 2)k - 1:** It is clear that  $d^{(2p-2)k-1}(0) = 0$ .

**(2p - 2)k:** We show that  $d^{(2p-2)k}(d^{(2p-2)k-1}\langle a_m \rangle) = (0, 0)$  for any sequence  $\langle a_m \rangle$ ,  $a_m \in \mathbb{Z}_{(p)}$ :

$$\begin{aligned}
\Psi_*(\Psi^*\langle a_m \rangle) &= \Psi_* \left\langle \begin{pmatrix} p^{\nu(k)+1}a_0 \\ a_1(r^{s(1)} - 1 + p^{\nu(k)+1}) + a_0 \\ \vdots \\ a_m(r^{s(m)} - 1 + p^{\nu(k)+1}) + a_{m-1} \\ \vdots \end{pmatrix} \right\rangle \\
&= \left\langle \begin{pmatrix} p^{2\nu(k)+1}a_0 \\ p^{\nu(k)+1}a_1(r^{s(1)} - 1 + p^{\nu(k)+1}) + p^{\nu(k)+1}a_0 \\ \vdots \\ p^{\nu(k)+1}a_m(r^{s(m)} - 1 + p^{\nu(k)+1}) + p^{\nu(k)+1}a_{m-1} \\ \vdots \end{pmatrix} \right\rangle \\
&= \Psi^* \left\langle \begin{pmatrix} p^{\nu(k)+1}a_0 \\ p^{\nu(k)+1}a_1 \\ \vdots \\ p^{\nu(k)+1}a_m \\ \vdots \end{pmatrix} \right\rangle = \Psi^*(\Psi_*\langle a_m \rangle).
\end{aligned}$$

Then

$$d^{(2p-2)k}(d^{(2p-2)k-1}\langle a_m \rangle) = (\Psi_*(\Psi^*\langle a_m \rangle) - \Psi^*(\Psi_*\langle a_m \rangle), 0) = (0, 0).$$

**$n = (2p - 2)k + 1$ :** Lastly, we verify that  $d^{(2p-2)k+1}(d^{(2p-2)k}(\langle a_m \rangle, \langle b_m \rangle)) = 0$  for any sequences  $\langle a_m \rangle, \langle b_m \rangle$ . This is immediate since the image of  $d^{(2p-1)k}$  is 0 in the second coordinate and  $d^{(2p-2)k+1}(a_m, 0) = 0$ .  $\square$

Before verifying that the cohomology is as expected, we examine the kernel of  $d^{(2p-2)k}$  more closely.

**Lemma 9.** *For all  $(\langle a_m \rangle, \langle b_m \rangle) \in \ker d^{(2p-2)k}$ ,  $p^{\nu(k)+1}|b_m$  for all  $m \in \mathbb{N}$ .*

*Proof.* If  $(\langle a_m \rangle, \langle b_m \rangle)$  is in the kernel, we know that

$$p^{\nu(k)+1}a_0 = p^{\nu(k)+1}b_0$$

and

$$p^{\nu(k)+1}a_m = (r^{s(m)} - 1 + p^{\nu(k)+1})b_m + b_{m-1} \quad \text{for all } m \geq 1.$$

Since  $r \in (\mathbb{Z}/p^2)^\times$ , we know  $r^{s(m)} - 1 = 0$  whenever  $s(m)$  is a multiple of  $p(p-1)$ . Now fix  $m \in \mathbb{N}$  and we will show that  $p^{\nu(k)+1}|b_m$ . Let  $\ell \in \mathbb{N}$ ,  $\ell > m$  such that  $r^{s(\ell)} - 1 = 0$ . Then  $p^{\nu(k)+1}a_\ell = p^{\nu(k)+1}b_\ell + b_{\ell-1}$  and thus,  $p^{\nu(k)+1}|b_{\ell-1}$ . Then since

$$p^{\nu(k)+1}a_q = (r^{s(q)} - 1 + p^{\nu(k)+1})b_q + b_{q-1}$$

it is clear that if  $p^{\nu(k)+1}|b_q$  then also  $p^{\nu(k)+1}|b_{q-1}$  for any  $q \geq 1$ . Thus since  $p^{\nu(k)+1}|b_\ell$  and  $\ell > m$ ,  $p^{\nu(k)+1}|b_m$  by induction.  $\square$

**Theorem 10.** *When  $k \neq 0$ ,*

$$H^n(C) = \begin{cases} \mathbb{Z}/p^{\nu(k)+1} & \text{if } n = (2p-2)k+1 \\ 0 & \text{else} \end{cases}$$

*Proof.* From the complex, it is immediate that  $H^t(C) = 0$  for all  $t$  that are not congruent to  $-1, 0, 1$ , or  $2$  modulo  $2p-2$ .

**$n = (2p - 2)k - 1$ :** Suppose  $\langle a_m \rangle$  is in  $\ker d^{(2p-2)k-1}$ . Then  $p^{\nu(k)+1}a_m = 0$  for all  $m \geq 0$ , and so  $a_m = 0$  for all  $m \geq 0$ . Thus  $\ker d^{(2p-2)k-1} = 0$  and so  $H^{(2p-2)k-1}(C) = 0$ .

**$n = (2p - 2)k$ :** Let  $(\langle a_m \rangle, \langle b_m \rangle) \in \ker d^{(2p-2)k}$ . This means

$$p^{\nu(k)+1}a_0 = p^{\nu(k)+1}b_0, \text{ so } a_0 = b_0,$$

and

$$p^{\nu(k)+1}a_m = (r^{s(m)} - 1 + p^{\nu(k)+1})b_m + b_{m-1} \text{ for all } m \geq 1.$$

By Lemma 9 we know  $p^{\nu(k)+1}|b_m$  for all  $m$ , and so we may write  $b_m = p^{\nu(k)+1}c_m$  for some  $c_m \in \mathbb{Z}_{(p)}$ . Thus,  $a_0 = b_0 = p^{\nu(k)+1}c_0$ , and for  $m \geq 1$  we may write

$$a_m = (r^{s(m)} - 1 + p^{\nu(k)+1})c_m + c_{m-1}.$$

Thus we see that all elements of the kernel are of the form

$$d^{(2p-2)k-1}(\langle c_m \rangle) = \left( \left\langle \begin{array}{c} p^{\nu(k)+1}c_0 \\ (r^{s(m)} - 1 + p^{\nu(k)+1})c_1 + c_0 \\ \vdots \\ (r^{s(m)} - 1 + p^{\nu(k)+1})c_m + c_{m-1} \\ \vdots \end{array} \right\rangle, \left\langle \begin{array}{c} p^{\nu(k)+1}c_0 \\ p^{\nu(k)+1}c_1 \\ \vdots \\ p^{\nu(k)+1}c_m \\ \vdots \end{array} \right\rangle \right)$$

Thus  $\ker d^{(2p-2)k} = \text{Im } d^{(2p-2)k-1}$  and  $H^{(2p-2)k}(C) = 0$ .

**$n = (2p - 2)k + 1$ :** Notice that

$$\ker d^{(2p-2)k+1} = \left\{ (\langle a_m \rangle, \langle b_m \rangle) \mid \begin{array}{l} p^{\nu(k)+1}b_0 = 0 \\ (r^{s(m)} - 1 + p^{\nu(k)+1})b_m + b_{m-1} = 0, \quad m \geq 1 \end{array} \right\}$$

where  $b_m \in \mathbb{Q}$  for all  $m \geq 0$ . Now if  $p^{\nu(k)+1}b_0 = 0$  then  $b_0 = 0$ . As  $r^{s(j)} - 1 + p^{\nu(k)+1} \neq 0$  for all  $j \geq 1$ , we conclude that  $(r^{s(j)} - 1 + p^{\nu(k)+1})b_j + b_{j-1} = 0$  implies that  $b_j = 0$  for all  $j \geq 1$ . Thus

$$\ker d^{(2p-2)k+1} = \left\{ (\langle a_m \rangle, \langle 0 \rangle) \mid a_m \in \mathbb{Z}_{(p)} \right\}.$$

We have used that  $\Psi^*(\langle b_m \rangle) = 0$  imply  $\langle b_m \rangle = 0$ .

For any  $(\langle a_m \rangle, \langle b_m \rangle) \in \mathbb{Z}_{(p)}^{\mathbb{N}} \oplus \mathbb{Z}_{(p)}^{\mathbb{N}}$  we have

$$d^{(2p-2)k}(\langle a_m \rangle, \langle b_m \rangle) = \left( \left\langle \begin{array}{c} p^{\nu(k)+1}(a_0 - b_0) \\ p^{\nu(k)+1}a_1 - (r^{s(1)} - 1 + p^{\nu(k)+1})b_1 - b_0 \\ \vdots \\ p^{\nu(k)+1}a_m - (r^{s(m)} - 1 + p^{\nu(k)+1})b_m - b_{m-1} \\ \vdots \end{array} \right\rangle, \left\langle \begin{array}{c} 0 \\ 0 \\ \vdots \\ 0 \\ \vdots \end{array} \right\rangle \right).$$

So if  $(\langle c_m \rangle, \langle 0 \rangle) \in \mathbb{Z}_{(p)}^{\mathbb{N}} \oplus \mathbb{Q}^{\mathbb{N}}$  is in  $\text{Im } d^{(2p-2)k}$ , then  $c_0$  is clearly divisible by  $p^{\nu(k)+1}$ . We will show that the converse is also true: if  $p^{\nu(k)+1}|c_0$ , then there exist sequences  $\langle a_m \rangle, \langle b_m \rangle \in \mathbb{Z}_{(p)}^{\mathbb{N}}$  such that  $d^{(2p-2)k}(\langle a_m \rangle, \langle b_m \rangle) = (\langle c_m \rangle, \langle 0 \rangle)$ .

Given any  $b_0$ , we may always select  $a_0$  so that  $p^{\nu(k)+1}(a_0 - b_0) = c_0$ . We will show that we can find  $a_m, b_m$ , and  $b_{m-1}$  so that

$$p^{\nu(k)+1}a_m - (r^{s(m)} - 1 + p^{\nu(k)+1})b_m - b_{m-1} = c_m$$

compatibly for all  $m \geq 1$ . As in the proof of Lemma 9, for any fixed  $m \in \mathbb{N}$ , we may choose the smallest value  $\ell > m$  such that  $r^{s(\ell)} - 1 = 0$ . Then if we take  $a_\ell = b_\ell$  and  $b_{\ell-1} = -c_\ell$  we have

$$p^{\nu(k)+1}a_\ell - (r^{s(\ell)} - 1 + p^{\nu(k)+1})b_\ell - b_{\ell-1} = c_\ell.$$

Now suppose we have defined  $a_q, b_q$ , and  $b_{q-1}$  so that

$$p^{\nu(k)+1}a_q - (r^{s(q)} - 1 + p^{\nu(k)+1})b_q - b_{q-1} = c_q.$$

If we then let  $a_{q-1} = b_{q-1}$  and  $b_{q-2} = (r^{s(q-1)} - 1)b_{q-1} - c_{q-1}$  we will obtain

$$p^{\nu(k)+1}a_{q-1} - (r^{s(q-1)} - 1 + p^{\nu(k)+1})b_{q-1} - b_{q-2} = c_{q-1}.$$

Again, inducting downwards from  $\ell$  shows that we can find values for  $a_m, b_m$  for any  $m$  such that  $d^{(2p-2)k}(\langle a_m \rangle, \langle b_m \rangle) = (\langle c_m \rangle, \langle 0 \rangle)$ .

$n = (2p - 2)k + 2$ : We need only show that  $d^{(2p-2)k+1}$  is surjective. We have

$$d^{(2p-2)k+1}(\langle a_m \rangle, \langle b_m \rangle) = \left\langle \begin{array}{c} p^{\nu(k)+1}b_0 \\ (r^{s(1)} - 1 + p^{\nu(k)+1})b_1 + b_0 \\ \vdots \\ (r^{s(m)} - 1 + p^{\nu(k)+1})b_m + b_{m-1} \\ \vdots \end{array} \right\rangle$$

where  $b_m \in \mathbb{Q}$  for all  $m \geq 0$ . Let  $\langle c_m \rangle \in \mathbb{Q}^{\mathbb{N}}$ . Set  $b_0 = \frac{c_0}{p^{\nu(k)+1}}$  and for all  $m \geq 1$ ,  $b_m = \frac{c_m - b_{m-1}}{r^{s(m)} - 1 + p^{\nu(k)+1}}$ . Then  $d^{(2p-2)k+1}(\langle a_m \rangle, \langle b_m \rangle) = \langle c_m \rangle$  and the map is surjective.  $\square$

## 7 Products and Massey Products

In this section we discuss the multiplicative structure of  $C$ , showing that it induces an injective multiplication  $H^{-(2p-2)k+1}(C) \otimes H^{(2p-2)k+1}(C) \rightarrow H^2(C)$  and that  $C$  has the appropriate Massey products.

### 7.1 Products

In this section we will prove the following:

**Proposition 11.** *The multiplication  $C^{-(2p-2)k+1} \otimes C^{(2p-2)k+1} \rightarrow C^2$  induces multiplication  $H^{-(2p-2)k+1}(C) \otimes H^{(2p-2)k+1}(C) \rightarrow H^2(C)$  given by*

$$\begin{aligned} \mathbb{Z}/p^{\nu(k)+1} \otimes \mathbb{Z}/p^{\nu(k)+1} &\longrightarrow \mathbb{Q}/\mathbb{Z}_{(p)} \\ a \otimes b &\longmapsto \frac{a}{p^{\nu(k)+1}} \frac{b}{p^{\nu(k)+1}} \end{aligned}$$

This will immediately give the following:

**Corollary 12.** *The multiplication  $H^{-(2p-2)k+1}(C) \otimes H^{(2p-2)k+1}(C) \rightarrow H^2(C)$  is injective.*

In order to prove Proposition 11, we examine the multiplication on  $C^*$ . The multiplication  $C^{-(2p-2)k+1} \otimes C^{(2p-2)k+1} \rightarrow C^2$  is of the form

$$\begin{array}{c} \text{Hom}_{\mathcal{B}}(E(1)_{*-n}E(1), E(1)_*E(1)) \otimes \text{Hom}_{\mathcal{B}}(E(1)_{*+n}E(1)E(1)_*E(1)) \\ \downarrow \\ \text{Hom}_{\mathcal{B}}(E(1)_*E(1), E(1)_* \otimes \mathbb{Q}) \end{array}$$

For  $f \in \text{Hom}_{\mathcal{B}}(E(1)_{*-n}E(1), E(1)_*E(1))$  and  $g \in \text{Hom}_{\mathcal{B}}(E(1)_{*+n}E(1)E(1)_*E(1))$  the composition on  $E(1)_*E(1)$ -comodules is

$$E(1)_*E(1) \xrightarrow{\Delta} E(1)_*E(1) \otimes E(1)_*E(1) \xrightarrow{f \otimes g} E(1)_{*-n}E(1) \otimes E(1)_{*+n}E(1) \xrightarrow{\mu} E(1)_*E(1)$$

To obtain the product in  $\text{Hom}_{\mathcal{B}}(E(1)_*E(1), E(1)_* \otimes \mathbb{Q})$ , we compose with  $q$ .

We translate this into a product on our sequence representations.

**Lemma 13.** *For sequences  $\langle a_m \rangle$  and  $\langle b_m \rangle$  representing  $\sum_{m \geq 0} a_m \Theta_m$  and  $\sum_{n \geq 0} b_n \Theta_n$  in  $E(1)^t E(1)$  and  $E(1)^s E(1)$  respectively, where  $t = (2p-2)k$  and  $s = (2p-2)\ell$ ,*

$$\sum_{m \geq 0} a_m \Theta_m \cdot \sum_{n \geq 0} b_n \Theta_n = \sum_{m+n=i} a_m b_n \Theta_m \Theta_n p^{N(i+k,m)-N(i,m)+N(i+\ell,n)-N(i,n)}$$

where  $N(i, k)$  are integers that depend on  $i$  and  $k$ .

*Proof.* Recall that when  $n = (2p-2)k$  we obtain the sequences using the equivalence

$$\text{Hom}_{\mathcal{B}}(E(1)_{*-n}E(1), E(1)_*E(1)) \cong E(1)^n E(1) = E(1)^0 E(1) \cdot v_1^k$$

So we consider

$$\begin{array}{ccc} E(1)^t E(1) \otimes E(1)^s E(1) & \longrightarrow & E(1)^{s+t} E(1) \\ \parallel & & \parallel \\ E(1)^0 E(1) \cdot v_1^k \otimes E(1)^0 E(1) \cdot v_1^\ell & \longrightarrow & E(1)^0 E(1) \cdot v_1^{k+\ell} \end{array}$$

for the product of elements from  $C^t$  and  $C^s$  where  $t = (2p-2)k$  and  $s = (2p-2)\ell$ . Since  $E(1)^0 E(1) = \{\sum_{m \geq 0} a_m \Theta_m\}$  where  $\Theta_m = \Theta_m(\Psi^r - 1)$ , we need to understand how  $\sum_{m \geq 0} a_m \Theta_m$  acts on  $v_1^i$ .



If  $t = 0$  then

$$\begin{aligned} \sum_{m \geq 0} a_m \Theta_m \cdot v_1^i &= \sum_{m \geq 0} a_m (r^{i(p-1)} - 1)(r^{i(p-1)} - r) \cdots (r^{i(p-1)} - r^{s(m)}) v_1^i \\ &= \sum_{m \geq 0} a_m p^{N(i,m)} v_1^i \end{aligned}$$

where  $N(i, m)$  is some integer depending on  $i$  and  $m$ . If  $t = (2p-2)k$  for  $k \neq 0$ ,

$$\begin{aligned} \sum_{m \geq 0} a_m \Theta_m \cdot v_1^i &= \left( \sum_{m \geq 0} a_m \Theta_m \cdot v_1^{i+k} \right) v_1^{-k} \\ &= \sum_{m \geq 0} a_m (r^{i(p-1)} - 1)(r^{i(p-1)} - r) \cdots (r^{i(p-1)} - r^{s(m)}) v_1^i \\ &= \sum_{m \geq 0} a_m p^{N(i+k,m)} v_1^i \end{aligned}$$

Applying this to the sum yields the product described in the lemma.  $\square$

**Corollary 14.** *The degree term in the sequence  $\langle a_m \rangle \cdot \langle b_n \rangle$  is  $a_0 b_0$ .*

*Proof.* From the definition we see that  $N(i, 0) = 0$  for any  $i$ , since  $v_1^0 = 1$ . Since the only way for  $\Theta_m \Theta_n = \Theta_0$  is to have  $m = n = 0$ , this proves the claim.  $\square$

*Proof of Proposition 11.* We saw in the homology computation of Theorems 7 and 10 that the homology in  $H^{(2p-2)k+1}(C)$  and  $H^2(C)$  is represented by the value of the index zero term in the sequences. Thus, to compute a product  $H^{-(2p-2)k+1}(C) \otimes H^{(2p-2)k+1}(C) \rightarrow H^2(C)$  we need only consider the multiplication  $C^{-(2p-2)k+1} \otimes C^{(2p-2)k+1} \rightarrow C^2$  on the index zero terms of sequences. By Corollary 14, if  $\langle a_n \rangle \cdot \langle b_n \rangle = \langle c_n \rangle$  then  $c_0 = a_0 b_0$ . Therefore if we pick any  $a \in H^{-(2p-2)k+1}(C) = \mathbb{Z}/p^{\nu(k)+1}$  and  $b \in H^{(2p-2)k+1}(C) = \mathbb{Z}/p^{\nu(k)+1}$ , we know multiplying them will yield the product in the quotient in  $H^2(C) = \mathbb{Q}/\mathbb{Z}_{(p)}$ . Explicitly, we first consider  $a$  and  $b$  as  $\frac{a}{p^{\nu(k)+1}} \in \mathbb{Q}/\mathbb{Z}_{(p)}$  and  $\frac{b}{p^{\nu(k)+1}} \in \mathbb{Q}/\mathbb{Z}_{(p)}$  respectively, in  $\mathbb{Q}/\mathbb{Z}_{(p)}$  and then multiply these representatives together.  $\square$

## 7.2 Massey Products

Here we calculate the Massey products.

**Proposition 15.** *Suppose that  $\gamma_k$  denotes an element of the cohomology  $H^{(2p-2)k+1}(C) \cong \mathbb{Z}/p^{\nu(k)+1}$  such that  $p\gamma_k = 0$ . Then the  $\gamma_k$ 's satisfy the following Massey product relation:*

$$\langle \gamma_i, p, \gamma_j \rangle = \gamma_{i+j}$$

*and the indeterminacy of this product is zero.*

*Proof.* We will compute the product directly using the definition of Massey product. The cohomology class  $\gamma_k$  must be a multiple of  $p^{\nu(k)}$ , and we can represent it by the cycle

$$a = \left( \left\langle \begin{array}{c} p^{\nu(k)+1}a_0 \\ 0 \\ 0 \\ \vdots \end{array} \right\rangle \left\langle \begin{array}{c} 0 \\ 0 \\ 0 \\ \vdots \end{array} \right\rangle \right)$$

where  $a_0$  is some value such that  $\nu(a_0) = 0$ . Choosing the analogous representative for  $\gamma_j$  gives the following cycles,  $a$ ,  $b$ , and  $c$ , representing  $\gamma_i$ ,  $p$ , and  $\gamma_j$  respectively:

$$a = \left( \left\langle \begin{array}{c} p^{\nu(k)+1}a_0 \\ 0 \\ 0 \\ \vdots \end{array} \right\rangle \left\langle \begin{array}{c} 0 \\ 0 \\ 0 \\ \vdots \end{array} \right\rangle \right) \quad b = p \quad c = \left( \left\langle \begin{array}{c} p^{\nu(k)+1}c_0 \\ 0 \\ 0 \\ \vdots \end{array} \right\rangle \left\langle \begin{array}{c} 0 \\ 0 \\ 0 \\ \vdots \end{array} \right\rangle \right)$$

where  $\nu(a_0) = \nu(c_0) = 0$ .

Now we choose

$$u = \left( \left\langle \begin{array}{c} a_0 \\ 0 \\ 0 \\ \vdots \end{array} \right\rangle \left\langle \begin{array}{c} 0 \\ 0 \\ 0 \\ \vdots \end{array} \right\rangle \right) \quad \text{and} \quad v = \left( \left\langle \begin{array}{c} -c_0 \\ 0 \\ 0 \\ \vdots \end{array} \right\rangle \left\langle \begin{array}{c} 0 \\ 0 \\ 0 \\ \vdots \end{array} \right\rangle \right)$$

where  $|u| = (2p-2)i$  and  $|v| = (2p-2)j$ . We can compute

$$d^{(2p-2)i}(u) = \left( \left\langle \begin{array}{c} p^{\nu(i)+1}a_0 \\ 0 \\ 0 \\ \vdots \end{array} \right\rangle \left\langle \begin{array}{c} 0 \\ 0 \\ 0 \\ \vdots \end{array} \right\rangle \right) = pa = (-1)^{1+|a|}a \cdot b$$

and

$$d^{(2p-2)j}(v) = \left( \left\langle \begin{array}{c} -p^{\nu(j)+1}c_0 \\ 0 \\ 0 \\ \vdots \end{array} \right\rangle \left\langle \begin{array}{c} 0 \\ 0 \\ 0 \\ \vdots \end{array} \right\rangle \right) = -pc = (-1)^{1+|b|}b \cdot c.$$

Therefore the Massey product  $\langle \gamma_i, p, \gamma_j \rangle$  can be computed as  $[(-1)^{1+|u|}u \cdot c + (-1)^{1+|a|}a \cdot v]$ . This gives us

$$- \left( \left\langle \begin{array}{c} a_0 \\ 0 \\ 0 \\ \vdots \end{array} \right\rangle \left\langle \begin{array}{c} 0 \\ 0 \\ 0 \\ \vdots \end{array} \right\rangle \right) \cdot \left( \left\langle \begin{array}{c} p^{\nu(k)+1}c_0 \\ 0 \\ 0 \\ \vdots \end{array} \right\rangle \left\langle \begin{array}{c} 0 \\ 0 \\ 0 \\ \vdots \end{array} \right\rangle \right) + \left( \left\langle \begin{array}{c} p^{\nu(k)+1}a_0 \\ 0 \\ 0 \\ \vdots \end{array} \right\rangle \left\langle \begin{array}{c} 0 \\ 0 \\ 0 \\ \vdots \end{array} \right\rangle \right) \cdot \left( \left\langle \begin{array}{c} -c_0 \\ 0 \\ 0 \\ \vdots \end{array} \right\rangle \left\langle \begin{array}{c} 0 \\ 0 \\ 0 \\ \vdots \end{array} \right\rangle \right)$$

which yields

$$\left( \left\langle \begin{array}{c} -2a_0c_0(p^{\nu(i)} + p^{\nu(j)}) \\ 0 \\ 0 \\ \vdots \end{array} \right\rangle \left\langle \begin{array}{c} 0 \\ 0 \\ 0 \\ \vdots \end{array} \right\rangle \right)$$

by our description of the multiplication in Section 7.1.

Now we can rewrite  $p^{\nu(i)} + p^{\nu(j)}$  as

$$p^{\nu(i)} + p^{\nu(j)} = p^{\min(\nu(i), \nu(j))} (1 + p^{\max(\nu(i), \nu(j)) - \min(\nu(i), \nu(j))}).$$

If  $i \neq j$  then  $\nu(i+j) = \min(\nu(i), \nu(j))$  so  $p^{\nu(i)+\nu(j)} = p^{\nu(i+j)}m$  where  $\nu(m) = 0$ . If  $i = j$  then  $\nu(i+j) = \nu(2i) = \nu(2) + \nu(i) = \nu(i)$ . Thus, in this case  $p^{\nu(i)+\nu(j)} = 2p^{\nu(i)} = 2p^{\nu(i+j)}$ .

Thus,

$$(-1)^{1+|u|}u \cdot c + (-1)^{1+|a|}a \cdot v = \left( \left\langle \begin{array}{c} -2a_0c_0m(p^{\nu(i+j)}) \\ 0 \\ 0 \\ \vdots \end{array} \right\rangle \left\langle \begin{array}{c} 0 \\ 0 \\ 0 \\ \vdots \end{array} \right\rangle \right)$$

where  $m$  is some value such that  $\nu(m) = 0$ . Thus we also have  $\nu(2a_0c_0m) = 0$  so this is an element of  $H^{(2p-2)(i+j)+1}(C)$  of order  $p$  which represents  $\gamma_{i+j}$ .

Finally, we note that the indeterminacy of the product is

$$\gamma_i H^{(2p-2)j}(C) \oplus \gamma_j H^{(2p-2)i}(C)$$

which is zero because the cohomology in each of those degrees is zero. □

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